

From canonical to nonautonomous solitons

Dun Zhao,^{1,2} Xu-Gang He,¹ and Hong-Gang Luo^{2,3,4}

¹*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China*

²*Center for Interdisciplinary Studies, Lanzhou University, Lanzhou 730000, China*

³*Key Laboratory for Magnetism and Magnetic materials of the Ministry of Education, Lanzhou University, Lanzhou 730000, China*

⁴*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China*

In this paper we show a systematical method to obtain exact nonautonomous soliton solutions of the nonautonomous nonlinear Schrödinger (NLS) equation named by Serkin, Hasegawa, and Belyaeva [Phys. Rev. Lett. **98**, 074102 (2007)]. We first perform the Painlevé analysis of the nonautonomous NLS equation and obtain a compatibility condition. Under this condition a general transformation is then found, which can reduce the nonautonomous NLS equation to the standard one. From this transformation, all solutions of the standard NLS equation can be converted into the corresponding solutions of the nonautonomous NLS equation. This builds a unified picture of the nonautonomous solitons and the canonical ones and deepens the understanding of the nonautonomous soliton dynamics. Finally, the fundamental bright and dark solitons are taken as examples to demonstrate explicitly its applications.

Introduction. – The solitary wave is a fundamentally nonlinear object in nature. It has been widely observed in various fields since 1834 [1]. The concept of soliton was first proposed by Zabusky and Kruskal [2] since they found in 1965 that the nonlinear solitary waves can pass through each other and preserve their shapes and speed after a collision, just like an elementary particle. This special feature of these solitary waves explained the recurrence phenomenon observed by Fermi, Pasta, Ulam, and Tsingou [3]. Since then, extensive theoretical and experimental investigations on the solitons have been made and some important applications of the solitons on the real world like the optical soliton communication [4] were also explored.

Why can the soliton behave like an elementary particle? As early as in 1870's, Boussinesq [5] and Rayleigh [6] have realized that the solitary water wave form results from the balance between the increase in local wave velocity associated with finite amplitude and the decrease associated with dispersion. In fact, this observation is also true to all solitons observed in various fields, namely, a localized nonlinear wave propagating in a medium can form a soliton when the effect of dispersion and that of nonlinearity is balanced. In addition, a dynamical balance between dissipation and nonlinearity can also stabilize a solitary wave [7]. Therefore, the general picture that the balance between different effects in a nonlinear system can result in a solitary wave provides a good opportunity to develop the concept of solitons in various complex situations.

As the first extension of the soliton concept, in 1976 Chen and Liu [8] found that the soliton can be accelerated in a linearly inhomogeneous plasma. At the same time the exact soliton solutions for the Korteweg-de Vries equation with varying nonlinearity and dispersion have also been found by Calogero and Degasperis [9]. Recently, the soliton dynamics for the discrete [10, 11] and

the generalized NLS equation with varying dispersion, nonlinearity, and dissipation or gain [12, 13, 14, 15, 16] have been extensively investigated. These solitons are apparently different from the classical solitons introduced by Zabusky and Kruskal [2], which propagate without energy dissipation or gain [17]. In 2007, Serkin, Hasegawa, and Belyaeva [18] introduced a concept of nonautonomous solitons, which are the exact solutions of the nonautonomous NLS equation with time- and/or space-dependent dispersion and nonlinearity. Distinguishing from the nonautonomous soliton, the classical one can be called as the canonical soliton. The various complex nonlinear optical media [4] and the Bose-Einstein condensates (BEC) tuned by the Feshbach resonances [19] provide many good grounds to study the dynamics of the nonautonomous solitons.

Two questions arise: what is the essential property of the nonautonomous solitons? What is the relationship between the nonautonomous solitons and the canonical ones? Some preliminary studies based on Lax pair [18] and similarity transformation [20] techniques indicated that under some integrability conditions the nonautonomous solitons remain the basic property of the canonical solitons but both amplitudes and speeds of the solitons vary with time and space. However, the second question is not yet clearly answered.

In this paper we focus on the second question and take a generalized NLS (or Gross-Pitaevskii) equation with varying dispersion, nonlinearity and harmonic external potential as an example. Using the Painlevé analysis we first study a compatibility condition under which the nonautonomous NLS equation might be completely integrable. Then, under this condition we try to construct a transformation which can reduce the nonautonomous NLS equation to a standard (or autonomous) NLS equation. As a result, this transformation builds a mapping between the canonical solitons and the nonautonomous

ones, which provides a systematical and significant way to study the dynamics of the nonautonomous solitons since the dynamics of the canonical solitons have been well-studied in more than hundred years.

Model and Painlevé analysis. – We start from a one-dimensional nonautonomous NLS equation

$$i \frac{\partial u(x, t)}{\partial t} + \varepsilon f(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + \delta g(x, t) |u(x, t)|^2 u(x, t) + V(t) x^2 u(x, t) = 0, \quad (1)$$

where $f(x, t)$ and $g(x, t)$ denote the time- and space-dependent dispersion and nonlinearity, respectively. ε and δ are constants which describe different systems to which the nonautonomous NLS equation can be applied. $V(t)$ represents a time-dependent confining harmonic external potential strength. These coefficients are usually assumed to be real.

Motivated by the connection between complete integrability and the Painlevé property of partial differential equations [21, 22], we perform the WTC test [21] to study possible integrability condition of Eq. (1). Following the Kruskal ansatz [23], $u(x, t)$, $v(x, t)$, $f(x, t)$, and $g(x, t)$ can be expanded on a non-characteristic singularity manifold $\varphi(x, t) = x + \phi(t)$, where $v(x, t) = u^*(x, t)$ is viewed as an independent function. The expansions read

$$\begin{cases} u(x, t) = (x + \phi(t))^{-p} \sum_{m=0}^{\infty} u_m(t) (x + \phi(t))^m, \\ v(x, t) = (x + \phi(t))^{-q} \sum_{m=0}^{\infty} v_m(t) (x + \phi(t))^m, \end{cases} \quad (2)$$

where $u_0 \neq 0, v_0 \neq 0$ are required and p and q are integers to be determined. Likewise,

$$\begin{cases} f(x, t) = \sum_{n=0}^{\infty} f_n(t) (x + \phi(t))^n, \\ g(x, t) = \sum_{n=0}^{\infty} g_n(t) (x + \phi(t))^n, \end{cases} \quad (3)$$

where $f_n(t) = \frac{1}{n!} \frac{\partial^n f(x, t)}{\partial x^n} \Big|_{x=-\phi(t)}$ and $g_n(t)$ is similar.

Inserting the above expansions into Eq. (1) and its complex conjugate equation, the leading order analysis leads to $p = q = 1$ and $2f_0(t) + g_0(t)u_0(t)v_0(t) = 0$. The recursion relations read

$$\begin{pmatrix} Q_m & g_0 u_0^2 \\ g_0 v_0^2 & Q_m \end{pmatrix} \begin{pmatrix} u_m \\ v_m \end{pmatrix} = \begin{pmatrix} F_m \\ G_m \end{pmatrix}, \quad (4)$$

where $Q_m = (m-1)(m-2)f_0 + 2g_0 u_0 v_0$ and the expressions of $F_m(G_m)$ are shown in Ref. [24]. The zeros of the matrix determinant in Eq. (4) give the resonance points $m = -1, 0, 3, 4$ of Eq. (1). While $m = -1$ and 0 lead to trivial results, the compatibility conditions under which Eq. (4) has a nontrivial solution are given by

$$m = 3: v_0(t) F_3 - u_0(t) G_3 = 0, \quad (5)$$

$$m = 4: v_0(t) F_4 + u_0(t) G_4 = 0. \quad (6)$$

These two conditions require that $f(x, t)$ and $g(x, t)$ are space-independent, i.e., $f(x, t) = f(t)$ and $g(x, t) = g(t)$. Furthermore, it is shown that Eq. (6) can be written as

$$\frac{g_{t,t}}{g} - \frac{2g_t^2}{g^2} + \frac{f_t^2}{f^2} - \frac{f_{t,t}}{f} + \frac{g_t}{g} \frac{f_t}{f} + 4\varepsilon f V = 0, \quad (7)$$

where the subscripts denote the time derivatives. This is a compatibility condition that Eq. (1) is Painlevé integrable. It is interesting to note that this condition is completely consistent with that obtained by Lax pair [18], but the space independence of dispersion and nonlinearity coefficients are required by the Painlevé analysis.

The complete integrability of Eq. (1) under the compatibility condition can be further confirmed if there exists a transformation that reduces Eq. (1) to the standard NLS equation

$$i \frac{\partial Q(X, T)}{\partial T} + \varepsilon \frac{\partial^2 Q(X, T)}{\partial X^2} + \delta |Q(X, T)|^2 Q(X, T) = 0. \quad (8)$$

Below we look for such a transformation in a general form of [25]

$$u(x, t) = Q(X(x, t), T(t)) e^{ia(x, t) + c(t)}, \quad (9)$$

where $X(x, t)$, $T(t)$, $a(x, t)$ and $c(t)$ are real functions and $u(x, t)$ and $Q(X, T)$ are required to be the solutions of Eqs. (1) and (8), respectively. Inserting Eq. (9) into Eq. (1) and comparing with Eq. (8), we obtain a set of differential equations, which have solutions under the condition Eq. (7),

$$a(x, t) = \frac{1}{4\varepsilon f(t)} \left(\frac{d}{dt} \ln \frac{f(t)}{g(t)} \right) x^2 + C_1 \frac{g(t)}{f(t)} x - C_1^2 \varepsilon \int \frac{g(t')^2}{f(t')} dt' + C_2, \quad (10)$$

$$X(x, t) = \frac{g(t)}{f(t)} x - 2C_1 \varepsilon \int \frac{g(t')^2}{f(t')} dt'. \quad (11)$$

$$T(t) = \int \frac{g^2(t')}{f(t')} dt' + C_3, \quad (12)$$

$$c(t) = \frac{1}{2} \ln \frac{g(t)}{f(t)}, \quad (13)$$

where C_1, C_2 , and C_3 are arbitrary constants and are set to be zero in the following discussions. The Painlevé integrability condition Eq. (7) together with the transformation Eqs. (9 - 13) consist of our central results of the present work.

An important consequence of these equations is that it provides a systematical way to find the exact solutions of the nonautonomous NLS equation (1), as shown in Fig. 1. For a given nonautonomous NLS equation, we first check if the coefficients satisfy the compatibility condition Eq. (7). If it is true, then the nonautonomous NLS equation can be reduced to the standard NLS equation (8). All exact solutions, including the canonical solitons,

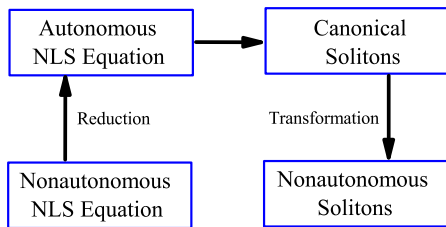


FIG. 1: Schematic diagram to obtain the nonautonomous soliton solutions of the nonautonomous NLS equation from the canonical solitons of the autonomous NLS equation.

of the standard NLS equation can thus be converted into the corresponding solutions of the nonautonomous NLS equation. In this sense, a canonical soliton can be viewed as a “seed” of the corresponding nonautonomous solitons under the compatibility condition Eq. (7), as emphasized on the title of the paper.

Some remarks are in order. i) If $f(t) = g(t)$ and $V(t) = 0$, the nonautonomous NLS equation Eq. (1) have the canonical soliton solutions (up to a phase) regardless of the explicit form of the time-dependent nonlinearity and dispersion. This is because in this case the balance between nonlinearity and dispersion is not broken down. In this sense the nonautonomous soliton is a quasi-canonical soliton. ii) When $g(t) \neq f(t)$, the original balance between nonlinearity and dispersion is broken down. In this case the canonical soliton deforms itself to build new balance between nonlinearity and dispersion. In this sense, the nonautonomous soliton is a deformed canonical soliton. The amplitude of the soliton will be scaled by the factor of $\sqrt{g(t)/f(t)}$, as shown by $c(t)$. iii) It is very interesting to note that the confining harmonic external potential is absent in the transformation equations. However, the presence of the potential affects the balance between nonlinearity and dispersion and builds a deep connection between the optical solitons and the matter-wave ones. Once a BEC soliton is formed in a confining potential, it can behave like an optical soliton, possibly can be controlled by appropriate nonlinearity and dispersion managements even without the external potential. iv) If $V(t) = 0$, the solitons can be quasi-canonical or deformed depending on if $f(t)$ is equal to $g(t)$ or not. On the contrary, if $V(t) \neq 0$, $f(t)$ must be unequal to $g(t)$. This leads to an important observation that there does not exist the canonical and even the quasi-canonical the matter-wave solitons under the compatibility condition Eq. (7) and will be discussed in detail elsewhere.

It is also helpful to mention some techniques to find the nonautonomous soliton solutions of the nonautonomous NLS equation in the literature. The Lax pair analysis is very useful in discussing integrability conditions [12, 16, 18]. The similarity transformation is usually limited to the very explicit transformation form, as shown

in Ref. [26]. Another similarity transformation reduced the nonautonomous NLS equation to a stationary NLS one has also been introduced [20]. Quite different from these techniques, the present work builds a connection between the nonautonomous NLS equation and its autonomous counterpart and provides a more systematical way to find solutions of the nonautonomous NLS equation. To explicitly show how the method works, below we consider the fundamental bright and dark soliton solutions of the nonautonomous NLS equation without the harmonic external potential.

Examples. – In this case, Eq. (7) has a solution

$$f(t) = g(t) \exp \left(-\alpha \int g(t') dt' \right), \quad (14)$$

where α is a constant and the transformation equations (10)-(13) become $a(x, t) = -\frac{\alpha}{4} \exp(G_\alpha(t))x^2$, $X(x, t) = \exp(G_\alpha(t))x$, $T(t) = \int dt' g(t') \exp(G_\alpha(t'))$, and $c(t) = (1/2)G_\alpha(t)$, where $G_\alpha(t) = \alpha \int_0^t g(t') dt'$.

When $\varepsilon = 1/2$ and $\delta = 1$ ($\delta\varepsilon > 0$), Eq. (8) has the fundamental canonical bright soliton solution $Q(X, T) = \text{sech}(X) \exp(iT/2)$ and when $\varepsilon = -1/2$ and $\delta = 1$ ($\delta\varepsilon < 0$), the fundamental dark soliton solution of Eq. (8) has the form of $Q(X, T) = \tanh(X) \exp(iT)$. Starting from these two solutions, we show the corresponding nonautonomous soliton solutions of Eq. (1) for four different cases: $g(t) = 1, \exp(t), \exp(-t)$, and $\cos(t)$, which represent constant, enhancement, suppression, and periodic modulations of nonlinearity, respectively. The corresponding dispersion modulations follow Eq. (14). It is noted that $\alpha = 0$ leads to $G_0(t) = 0$, which is trivial up to a phase, as mentioned above. A nonzero α has nontrivial results and without loss of generality we take $\alpha = 1$ below. In Fig. 2 and Fig. 3 we explicitly present the nonautonomous bright solitons and the nonautonomous dark solitons for four different nonlinearity modulations, respectively. For comparison, we also plot the canonical solitons, as shown in Fig. 2(a) and Fig. 3(a), respectively.

Fig. 2(b) and (c) show that the canonical bright soliton becomes more and more either sharper or broader depending on enhancement or suppression of the nonlinearity. When the nonlinearity keeps unchange and the dispersion is suppressed, the canonical bright soliton also becomes more and more sharper, as shown in Fig. 2(e). This can be understood by the fact that the soliton is due to the balance between dispersion and nonlinearity. Most interesting case is Fig. 2(d), where the nonlinearity modulation is periodic. As a result, the canonical bright soliton is also modulated periodically. These all results indicate that the nonautonomous bright soliton and its canonical counterpart has a close relationship. In all cases, it is easy to check that the integrations of $\int |u(x, t)|^2 dx$ keep unchange in time. The similar analysis is also true to the nonautonomous dark solitons, as

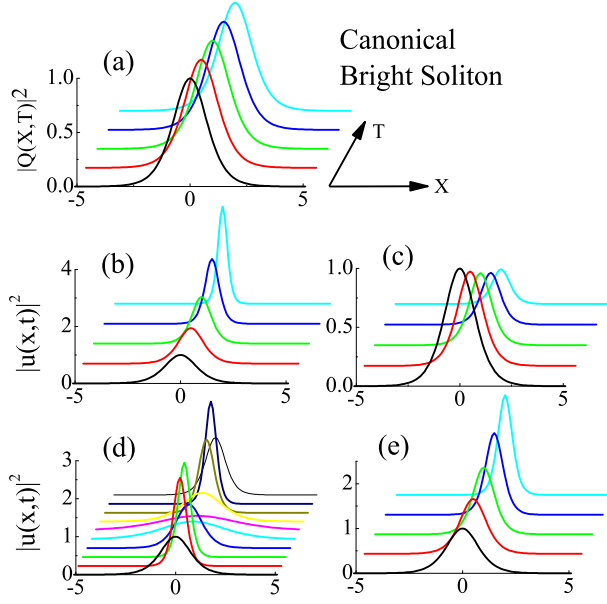


FIG. 2: (a) The canonical bright soliton of Eq. (8) with $\varepsilon = 1/2$ and $\delta = 1$ and the corresponding nonautonomous bright solitons of Eq. (1) for different nonlinearity modulations: (b) $g(t) = \exp(t)$, (c) $g(t) = \exp(-t)$, (d) $g(t) = \cos(t)$, and (e) $g(t) = 1$. In all cases $\alpha = 1$.

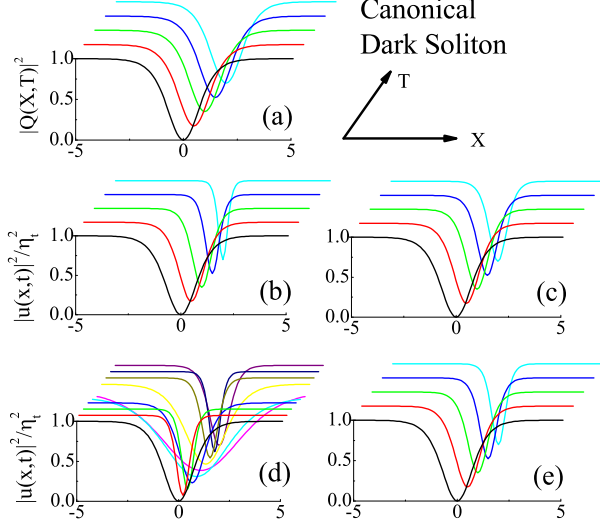


FIG. 3: The canonical dark soliton of Eq. (8) with $\varepsilon = -1/2$ and $\delta = 1$ and the corresponding nonautonomous dark solitons of Eq. (1) for different nonlinearity modulations same as those in Fig. 2. For clarity, the amplitude of the nonautonomous dark solitons is normalized by $\eta_t = \exp(\frac{1}{2}G_1(t))$.

shown in Fig. 3. These results shed light on the understanding of the nonautonomous soliton dynamics and provide an exact way to make a dispersion and/or nonlinearity management of solitons. It is expected to have a realistic application to the optical soliton communication technologies and the matter-wave soliton dynamics.

However, the presence of dissipation and/or gain in these systems will complicate the present analysis and a non-trivial extension is needed.

Finally, it should be pointed out that the present analysis can also be applied to all exact solutions of Eq. (8), including the multi-soliton cases. This provide a systematical way to study the dynamics of the nonautonomous NLS equation.

In conclusion, based on the Painlevé analysis and a general transformation obtained we have reduced the nonautonomous NLS equation to the standard one. This provide a systematical and significant way to obtain the nonautonomous soliton solutions of the nonautonomous NLS equation from its autonomous counterpart, i.e., the canonical solitons. The nonautonomous solitons obtained result from a balance between dispersion, nonlinearity, and/or an external potential applied, just like the canonical solitons. This result builds a unified picture of the nonautonomous and the canonical solitons and provides an effective way to control the nonautonomous soliton dynamics.

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$$F_m = \frac{-i(u_{m-2,t} + (m-2)\phi_t)}{\sum_{k=1}^m (m-k-1)(m-k-2)f_k u_{m-k}} - g_m u_0^2 v_0 - g_0 v_0 \sum_{n=1}^{m-1} u_{m-n} u_n - g_0 \sum_{n=1}^{m-1} \sum_{k=0}^n v_{m-n} u_{k-n} u_k - \sum_{n=1}^{m-1} \sum_{k=0}^n \sum_{l=0}^k g_{m-n} v_{n-k} u_{k-l} u_l - V x^2 u_{m-2}$$
 and G_m can be obtained by first interchanging u and v and then taking its complex conjugate.
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